Discussion of:

"A Theory of Arbitrage Free Dispersion"

by

Piotr Orlowski, Andras Sali, and Fabio Trojani

Caio Almeida

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Hansen and Jagannathan (HJ, 1991) use data on assets to verify if model implied SDFs (IMRS) present enough variance to match risk premia.

In their view, a model has a static representation given by its implied SDF.

This SDF could be either a function of observables, parametric, or simply have its moments calculated from equilibrium conditions.

Numerous generalizations of HJ (1991) were proposed:

- Accounting for conditional (GHT, 1990) and higher order moments (Snow, 1991, Bansal and Lehmann, 1997) to build new measures of misspecification.

- Accounting for SDF dynamics in addition to its implied moments Alvarez and Jermann (AJ, 2005).
Testing APMs critically depends on which observed data and on which measure of misspecification are adopted:

- HJ (1991) adopted only a stock index and a bond to derive their variance bound:
  - Tests if model SDF captures risk premium.

- Alvarez and Jermann (2005) include a long term bond to derive entropy bounds for the permanent component of the SDF.
  - Show the importance of having APMs with a volatile persistent SDF.

- Backus Chernov and Zin (BCZ, 2014) use whole term structure of bonds with horizon dependent entropy.
  - New measure accounts for pricing kernel dynamics when testing an APM.
Backus, Boyarchenko and Chernov (BBC, 2015) look at different markets (equities, bonds, currencies) introducing co-entropy between pricing kernel and returns.

- New measure is related to both risk premium and pricing kernel dynamics.

This paper: Unified methodology capturing all the above mentioned generalizations of HJ (1991):

- Accounts for higher moments (Snow, 1991), SDF persistence (AJ, 2005), SDF dynamics (BCZ, 2014), and interactions between SDF and cashflows (BBC, 2015).

- Beautiful theory on the joint CGF of SDF and returns.

In my view, it is an important contribution to the literature of asset pricing!

Authors need to show how the generality of the theory can bring new insights on empirical tests of APMs.
A Graphical Perspective in Testing Asset Pricing Models

Different measures of dispersion:

- Variance (HJ, 1991)
- KLIC (STUTZER, 1995)
- Entropy (BANSAL & LEHMANN, 1997)
- Higher moments (SNOW, 1991)
- CRESSIE-READ discrepancies (ALMEIDA & GARCIA, 2011)
- Co-entropy (BACKUS, CHEARNOV, 2015)

TESTING APMS

(RISK PREMIUM)

(SDF PERSISTENCE)
(Alvarez and Jermann, 2005)

(SDF DYNAMICS)
(MULTIPLE HORIZON ENTROPY)

BACKUS, CHEARNOV AND ZIN (2014)

GENERALIZATIONS: BAHSHE & CHAPIYO (2012, 2014)
Central Ingredients of the Methodology

- Start with a joint CGF of pricing kernel and returns:
  \[ K_{MR}(m, r) = \log E[M^m R^r]. \]

- It is a convex function on \((m, r)\).

- Use Jensen’s inequality in a smart way to provide (upper and, or lower) bounds for this CGF on regions of \((m, r)\) where it is not ”observed”.

- Here observation of CGF can appear in three forms:
  - No-arbitrage restrictions. Ex: \(E(M \ast R) = 1\) implies
    \[ K_{MR}(1, 1) = \log(E[M^1 R^1]) = 0. \]
  - Statistical observations. Ex: Returns \(R\) are observed implies
    \[ K_{MR}(0, r) = \log(E[R^r]) \text{ is observed, assuming that we can accurately estimate } E[R^r]. \]
  - Normalizations. Ex: Martingale component of pricing kernel has mean 1 implies
    \[ K_{M_p M_T R}(1, 0, 0) = \log E[M_p] = 0. \]
Jensen’s inequality: If $\phi(\cdot)$ is a convex function and $X$ a RV:
$$E(\phi(X)) \geq \phi(E(X))$$

Here the RVs will be the exponents $(m, r)$ of $M, R$ in the CGF!

The trick relies in choosing smart prior probability distr. $\pi$ on suitably chosen points $(m, r)$ and apply the inequality above:

$$E_\pi[K_{MR}(m, r)] \geq K_{MR}(E_\pi[(m, r)]) = K_{MR}(\bar{m}, \bar{r}) = \log(E[M^{\bar{m}}R^{\bar{r}}]) \tag{1}$$

When prior probability $\pi$ is chosen such that $(\bar{m}, \bar{r})$ are not observed, and $\pi$ has support on $O_{K_{MR}}$, we have a family of upper bounds for $E[M^{\bar{m}}R^{\bar{r}}]$.

When prior $\pi$ is chosen such that $(\bar{m}, \bar{r}) \in O_{K_{MR}}$ observed, and $\pi$ has support on $O_{K_{MR}} \cup (m^*, r^*)$, we have a family of lower bounds for $E[M^{m^*}R^{r^*}]$. 
Central Ingredients of the Methodology - Example

- Since we control the priors we can, for instance, set in (1) \( \bar{r} = 0 \) to get upper bounds for any moment of the SDF.

- Choose \( \pi(1, 1) = \alpha \) and \( \pi(0, -\frac{\alpha}{1-\alpha}) = 1 - \alpha \), to obtain \((\bar{m}, \bar{r}) = (\alpha, 0)\) and the following bound:
  \[
  E_{\pi}(K_{MR}(m, r)) \geq K_{MR}(E_{\pi}((m, r))) = K_{MR}(\bar{m}, \bar{r}) = \log(E[M^\alpha])
  \]

- Is this bound tight? Since the prior distr. is free we have to work to show it...

- For \( \alpha \in [0, 1] \ ((-\infty, +\infty)) \), Almeida and Garcia (AG, 2011) provide a tight lower bound for \( E[M^\alpha - E(M)^\alpha] \) based on maximization of a dual portfolio problem.

- OST (2015) rewrite their problem as a MD problem and use result from AG (2011) to obtain tight bounds without having to optimize over prior distributions.
Questions: Tight Bounds

- The authors adapt (AG, 2011) dual tight bounds to obtain tightness on a variety of higher moment bounds, including $M^\alpha$, $(M^T)^\alpha(M^P)^{1-\alpha}$, multi-period and multi-market...

- In some cases, however, adaptation of AG (2011) bounds is not available.

- For instance, how should we choose prior distributions to obtain tight bounds in general combinations of $(M, R)$ when both $\bar{m} \neq 0$ and $\bar{r} \neq 0$: 

$$E_\pi[K_{MR}(m, r)] \geq K_{MR}(E_\pi[(m, r)]) = K_{MR}(\bar{m}, \bar{r}) = \log(E[M^{\bar{m}}R^{\bar{r}}])$$

- Will have to deal, from an operational viewpoint, with the minimum convex extension of the CGF $(K^U(m, r))$ on the convex hull of $O_{K_{MR}}$.

- How difficult is to obtain it?
Tight Bounds II

**Bounds Tightness**

1. Priors between $K_{HR}(0, 1)$ and $K_{HR}(1, 0)$:
   \[ \lambda (0, 1) + (1-\lambda) (1, 0) = (1-\lambda, \lambda) \]

2. Priors between $K_{HR}(0, 0)$ and $K_{HR}(1, 1)$:
   \[ \lambda (0, 0) + (1-\lambda) (1, 1) = (1-\lambda, 1-\lambda) \]

3. Priors between unobserved $P_1$ and $P_2$:
   \[ \gamma P_1 + (1-\gamma) P_2 = \]
   \[ \gamma (0, 0) + (1-\gamma) (1-\beta) (1, 0) + (1-\gamma) (1-\beta) (1, 1) = (\gamma (1-\beta) + (1-\gamma) \beta, 1-\gamma) \]

Jensen's Inequality

\[ K_{HR}(1-\gamma, \gamma) < \gamma K_{HR}(0, 0) + (1-\gamma) K_{HR}(1, 1) \]

\[ K_{HR}(1-\gamma, \gamma) < \gamma K_{HR}(1, 0) + (1-\gamma) K_{HR}(1, 1) \]

\[ K_{HR}(\delta(1-\gamma) + (1-\delta) \beta, 1-\gamma) < \delta(1-\beta) K_{HR}(0, 0) + (1-\delta) K_{HR}(1, 1) \]

\[ K_{HR}(\delta(1-\gamma) + (1-\delta) \beta, 1-\gamma) < \delta(1-\beta) K_{HR}(1, 0) + (1-\delta) K_{HR}(1, 1) \]
Bound Tightness

If in the picture above we choose \( \alpha = \beta = \gamma = \frac{1}{2} \), we obtain three different bounds for \( K_{MR}(0.5, 0.5) \):

- \( K_{MR}(0.5, 0.5) \leq 0.5 * \log(B) + 0.5 * E(R) \)
- \( K_{MR}(0.5, 0.5) \leq 0 \)
- \( K_{MR}(0.5, 0.5) \leq 0.25 * \log(B) + 0.25 * E(R) \)

To obtain the tightest bound, we can show that we have to search, for any finite \( n \), for the minimum value of

\[ \alpha_1 * K_{MR}(m_1, r_1) + \alpha_2 * K_{MR}(m_2, r_2) + \ldots + \alpha_n * K_{MR}(m_n, r_n), \]

with \( \alpha_i \geq 0 \), probabilities, \( (m_i, r_i) \in O(K_{MR}) \),

\[ \sum_{i=1}^{n} \alpha_i * (m_i, r_i) = (0.5, 0.5). \]
Testing APMs: How much addressed in OST (2015)?

- Choice of **basis assets** and **measure of misspecification** are fundamental ingredients.

- What are relevant questions?
  
  1. Identify sources of model misspecification. OST: good job but how different from BCZ (2014), BBC (2015)?
  
  2. How to correct misspecification with respect to observed (adopted) basis assets?
     - Hansen and Jagannathan (HJ, 1997): Model is corrected by a linear combination of basis assets.
     - Almeida and Garcia (2012): Model is corrected by a hyperbolic function of a linear combination of basis assets.
     - Ghosh, Julliard and Taylor (2013): CCAPM is corrected by exponential (hyperbolic) funct. of a linear combination of basis assets.
     - Generalization of HJ (1997) distance?
3. Given a set of models, how to rank them based on your methodology?

4. Is it possible to modify the proposed measures to estimate a model?

In this case, estimation should be based on a continuum of upper and lower restrictions in the model CGF:

\[
\min_{p \in M_{SDF}} E\{\psi(1+p-y(\theta))\}, \text{s.t.} K_L^L(m, r) \leq K_{MR}^{1+p-y(\theta)}(m, r) \leq K_U^U(m, r)
\]

Convex minimization problem with convex constraints.

Chernozukov, Hong, Tamer (2007), Chernozukov, Kocatulum, Menzel (2012), Chernozukov, Lee, Rosen (2013) - Framework to estimate models with many moment inequality, intersection bounds...
Empirical Section: LRR and other models

- From an empirical viewpoint, it is important to distinguish your work from BCZ (2014), Bakshi and Chabi-Yo (2014)...

- Which new insights come with the methodology?
  - Are there tests that capture new dimensions not analyzed in previous papers?
  - Simpler APMs might offer more transparency on how much the generality of the methodology can contribute.
  - Also, a comparison of your results with the entropic benchmark cases of BCZ(2014) and BBC(2015) would be nice.
Empirical Section: LRR and other models II

- Some suggestions:
  - Go back to the Vasicek (1977) example in BCZ (2014) and explore horizon dependence with your generalized entropy.
  - Explore the generalized co-entropy bounds with the BY (2004) LRR model.
  - Or explore co-entropy bounds with a LRR model with disappointment aversion by Bonomo, Garcia, Meddahi and Tedongap (2011).
  - Disappointment aversion makes the model more robust: Better predictability results, EIS not restricted to be greater than 1, slightly less persistent consumption growth and consumption volatility.
Conclusions

- Very nice paper with a general and elegant theory of dispersion.

- Many interesting applications still to come.