Discussion of:
”Robust Preference Expansions”
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Solving DSGE models with misspecification and recursive utilities

- Dynamic equilibrium model with multiple agents with recursive utilities and aware of model misspecification.

- For each agent, model misspecification introduces a distorted worst-case probability that modifies the solution.

- Probabilities depend on the agent continuation value $V_{i+1}^i$ and a parameter of the degree of misspecification ($\theta_i$).

- Proposed solutions capture qualitative features (impulse response, risk premia) that standard solutions only capture if they go one order higher in the approximation.

- Combines elements of the literatures on robust control and model misspecification.
Robustness versus Recursive Preferences

- Recursive methods were tools in modern control theory years before (most) scientists showed concern about misspecification (Hansen and Sargent, 2008).

- Interesting venue: Try to disentangle the roles of misspecification (robustness) and recursive utilities.

- Redesign the problem to have one agent considering misspecification...

- ...choosing from a set of probability measures that help the model to satisfy the Euler equations implied in equilibrium:

\[
\hat{M}_{t+1}^{KLIC} = \arg \min E_t[M_{t+1} \log (M_{t+1})]
\]

s.t. \( E_t(M_{t+1} \hat{g}(x_{t+1}, z_{t+1}, q, ...)) = 0, E_t(M_{t+1}) = 1 \).

- where \( \hat{g}_{t+1}(., q) \) represents excess returns in the Euler equations.
Robustness versus Recursive Preferences, Continues...

- The minimum entropy measure
  \[ \hat{M}_{t+1}^{KLIC} = \exp\left(\frac{-\lambda(q) \hat{g}(q)}{E_t[\exp(-\lambda(q) \hat{g}(q))]}\right) \]

- can be used to distort model solution as in the paper.

- Choice of function \( \lambda(q) \) that correctly scales the problem:
  \[ \lambda(q) = \frac{\tilde{\lambda}}{q} \]


- Slightly different interpretation from the paper and from the framework of Hansen and Sargent (2008):
  - Chooses measure that corrects model pricing with smallest entropy versus (in the paper) measure that provides worst utility for agent with smallest entropy.

- Approach from literature on model misspecification with nonparametric likelihood (Kitamura, 2000).
Interpretation of limiting economies when $q \downarrow 0$

- Agent $i$ solves the following optimization problem with misspec. parameter $\theta_i$:

$$V_t^i = \min_{M_{t+1}^i, E_t(M_{t+1}^i)=1} u_t^i + q\theta_i E_t[M_{t+1}^i \log(M_{t+1}^i)] + E_t[M_{t+1}^i V_{t+1}^i]$$

- Scaling $\theta_i$ to $q\theta_i$ and sending $q$ to zero gives a desired result: Misspecification has a first order risk-premium.

- But what happens with the set of alternative measures that adjust misspecification in the limiting economy?

- Let us take the more general case where $\theta_i$ is replaced by $\theta_i(q)$ and study three cases for limiting economies:

  - i) $\theta_i$ is kept constant, case analyzed in the paper. ii) $\theta_i(q) \approx \frac{1}{q}$, and iii) $\theta_i(q) \cdot q \uparrow \infty$.

- Case i) has invariant set of measures. Cases ii) and iii) converge to the rational expectations model.
Interpretation of limiting economies when $q \downarrow 0$

- Scaling keeps approxim. constant the relative entropy between worst probability distortion and benchmark model when $q$ is small:
  \[
  M_{t+1}^i = \frac{\exp(-\frac{1}{\theta_i q} V_{t+1}^i)}{E_t[\exp(-\frac{1}{\theta_i q} V_{t+1}^i)]} \approx \frac{\exp(-\frac{1}{\theta_i} V_{1,t+1}^i)}{E_t[\exp(-\frac{1}{\theta_i} V_{1,t+1}^i)]}
  \]

- Expanding $V_{t+1}^i(q)$ in series shows that agent $i$ solves:
  \[
  q \cdot \min_{M_{t+1}^i, E_t(M_{t+1}^i)=1} \theta_i E_t[M_{t+1} \log(M_{t+1})] + E_t[M_{t+1}(\frac{V_{0,t+1}^i}{q} + V_{1,t+1}^i + \tilde{V}_{t+1}^i(q))]
  \]
  
- And that in the limit, this becomes
  \[
  \min_{M_{t+1}^i, E_t(M_{t+1}^i)=1} \theta_i E_t[M_{t+1} \log(M_{t+1})] + E_t[M_{t+1} V_{1,t+1}^i]
  \]

- Scaling shifts the importance on the optim. problem (and at the probability distortion) from $V_{0,t+1}^i$ to $V_{1,t+1}^i$.
  
  - Important because $V_{0,t+1}^i$ is assumed to be deterministic. Therefore, without scaling there would be no distortion.

- Behavior of set of alternative measures in the limit? Const!
Interpretation of limiting economies when $q \downarrow 0$

- The set doesn’t vary with $q$, being always equal to the set of probability measures that are constrained by $\theta_i$.
  - $q$ can be factorized at the agent optimization problem.
  - Scaling compensates changes in the structure of the continuation value as a function of $q$.
- What if agents learn and $q\theta_i(q)$ is kept constant? That is, for each $q$-economy, agent’s parameters $\theta(q)$ are adjusted.
  - Factorizing $q$ we see that agent $i$ solves a problem more and more restricted by $\theta_i(q) \uparrow \infty$.
  - In such case, the set of alternative measures becomes empty because it is too costly to change measure.
Robustness of the Robust Approach? The Cressie Read Family of Discrepancies

- Entropy is a useful and important quantity to measure the degree of model misspecification.

- Frequently adopted in other science fields and guarantees a positive worst case distortion probability. But...

- ...there are other types of discrepancies to measure the degree of model misspecification.

- Among them Empirical Likelihood and the Cressie Read family of discrepancies (Kitamura, 2006).

- Used in the econometric literature as a one-step alternative to GMM
  - Both when estimating correctly specified models and to measure misspecification of asset pricing models.
Robustness of the Robust Approach?

- Defining the new optimization problem of agent $i$:
  
  $$
  V^i_t = \min_{M^i_{t+1}, E_t(M^i_{t+1})=1} u^i_t + q\theta_i E_t[\phi(M_{t+1})] + E_t[M_{t+1}V^i_{t+1}]
  $$

  where $\phi(m) = \frac{m^{\gamma+1}-1}{\gamma(\gamma+1)}$.

- Solution is a hyperbolic function:
  
  $$
  M^{CR}_{t+1} = \frac{(1-\gamma \frac{\theta_i}{V^i_{t+1}}) \frac{1}{\gamma}}{E_t[(1-\gamma \frac{\theta_i}{V^i_{t+1}})^{\frac{1}{\gamma}}]}
  $$

- Different choices of $\gamma$ will imply measures that will distort differently the dynamics of the approximation solution.

- $\gamma = 0$ captures the entropy case. For $\gamma < -1$ the implied worst case measures may have extreme values since they are sensitive to higher moments of the continuation values.

- Such extreme values might help to increase even more risk-premia under the benchmark model.
Typical in the paper is: Differentiating with respect to $q$ expressions such as:

$$E_t[exp\left(-\frac{1}{\theta_i} (V_{1,t+1}^i + \frac{q}{2} V_{2,t+1}^i + \frac{q^2}{6} V_{3,t+1}^i)\right)]$$

by switching differential operator with the expectation and differentiating inside the conditional expectation.

Such exchange of operations demands integrability conditions (or monotonicity) on the R.V.s $V_{j,t+1}^i, j = 1, 2, 3$.

However, these random variables are still not known when solving the approximation. How to proceed then?

Assume that they are all limited by an integrable random variable and apply the dominated convergence theorem.

Caution: Implied worst case SDF contains exponentials of chi-squared R.V.s. Laplace transform might not exist.